TWO SUM THEOREMS FOR TOPOLOGICAL SPACES

BY

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ABSTRACT

Two general criterias for the union of topological spaces of a given type to be also of the same type, are presented.

In the present note, we have proved two general sum theorems for certain important classes of topological spaces. Two sum theorems proved by R. E. Hodel [3, theorems 1 and 2] follow as special cases of these theorems. For the class of paracompact spaces, these theorems improve the results of H. Tamano [14, theorem 2], Y. Katuta [4, theorem 1], and S. Hanai and A. Okuyama [2, theorem 3]. For the class of metrizable spaces, these, theorems improve the results of Y. Katuta [4, theorem 5] and A. H. Stone [13, theorem 3]. These sum theorems offer new results for certain important classes of spaces such as the class of normal spaces, the class of collectionwise normal spaces, the class of normal and $\mathscr{M}$-paracompact spaces, the class of pointwise paracompact spaces and the class of stratifiable spaces. Incidentally, we have obtained a result which improves results of E. Michael [7] and K. Morita [9].

To prove our first theorem, we need the following result of J. Mack [6]:

"A space is paracompact iff every directed open covering (A covering $\mathscr{V}$ is said to be directed if it is directed by set inclusion, that is, given $U, V \in \mathscr{V}$ there exists $W \in \mathscr{V}$ such that $U \cup V \subseteq W$) has a locally-finite closed refinement."

THEOREM 1. *Let $\{F_\alpha : \alpha \in \Lambda\}$ be a locally-finite, closed covering of a space X such that F_α is paracompact for each α. Then X is paracompact.*

PROOF. Let $\mathscr{V} = \{U_\delta : \delta \in \Delta\}$ be any directed open covering of X. Then for each α, $\{U_\delta \cap F_\alpha : \delta \in \Delta\}$ is a directed, relatively open covering of F_α which is paracompact. Therefore, there exists a locally-finite (in F_α and hence also in X), closed (in F_α and hence also in X) refinement $\{B_\beta : \beta \in I^\alpha\}$ of $\{U_\delta \cap F_\alpha : \delta \in \Delta\}$. Now, let $\mathscr{V} = \{B_\beta : \beta \in I^\alpha, \alpha \in \Lambda\}$. Then, it can be verified that $\mathscr{V}$ is locally-finite,

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closed refinement of $\mathscr{U}$. Thus every directed open covering of X has a locally-finite, closed refinement and therefore X is paracompact.

COROLLARY 1. (S. Hanai and A. Okuyama [2]). If $\{G_\alpha : \alpha \in \Lambda\}$ is a locally-finite, open covering of a space X such that $\bar{G}_\alpha$ is paracompact for each $\alpha \in \Lambda$, then X is paracompact.

Now we proceed to prove our two main general theorems. Denote by $\wp$, a weakly hereditary topological property (that is, a property which when possessed by the space is also possessed by every closed subspace of it) satisfying the following following:

'If $\{F_\delta : \delta \in \Delta\}$ is a locally-finite, closed covering of X such that each F_δ has property $\wp$, then X has property $\wp$.

H. Tamano [14] calls a family $\{A_\alpha : \alpha \in \Lambda\}$ linearly locally-finite if there is a linear ordering '$<$' of the index set Λ such that for each $\alpha \in \Lambda$, the family $\{A_\beta : \beta < \alpha\}$ is locally-finite. Y. Katuta [4] restricted this definition by requiring each $\{A_\beta : \beta < \alpha\}$ to be locally-finite at each point of U_α and called such a family order locally-finite.

Obviously, every linearly locally-finite family is order locally-finite but not conversely. Also, every σ-locally-finite family is linearly locally-finite but not conversely.

THEOREM 2. *Let $\mathscr{V}$ be an order locally-finite open covering of a space X such that the closure of each member of $\mathscr{V}$ possesses the property $\wp$. Then X possesses $\wp$.*

PROOF. Let $\mathscr{V} = \{V_\alpha : \alpha \in \Lambda\}$ where Λ is a linearly ordered index set such that $\{V_\beta : \beta < \alpha\}$ is locally-finite at each point of V_α for each α. For each α, let $U_\alpha = \bar{V}_\alpha \sim \cup\{V_\beta : \beta < \alpha\}$ and let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$. Obviously, each element of $\mathscr{U}$ is closed and has property $\wp$. Using a technique similar to the one due to Katuta [5], one can show that $\mathscr{U}$ is a locally-finite cover of X. Thus X must have property $\wp$.

COROLLARY. (R. E. Hodel [3]). Let $\mathscr{V}$ be a σ-locally-finite open covering of a space X such that the closure of each member of $\mathscr{V}$ possesses the property $\wp$. Then X possesses $\wp$.

REMARK. Since paracompactness is weakly hereditary, it follows from theorem 1 that theorem 2 is true for $\wp =$ paracompactness. This improves results of H. Tamano [14, theorem 2] and Y. Katuta [4, theorem 1] as mentioned in the introduction.

THEOREM 3. *If X is a regular space, $\mathscr{V}$ is an order locally-finite open covering of X each member of which possesses the property $\wp$ and if the frontier of each member of $\mathscr{V}$ is compact, then X possesses the property $\wp$.*

PROOF. Omitted, since it is similar to the proof of the following corollary in [3] due to Hodel.

COROLLARY. (R. E. Hodel [3]). If $\mathscr{V}$ is a σ-locally-finite open covering of a regular space X, each member of which possesses $\wp$ and if the frontier of each member of $\mathscr{V}$ is compact, then X possesses $\wp$.

REMARK. J. Nagata [11] has shown that a space which is the union of a locally-finite family of closed, metrizable sets is metrizable. Hence theorem 3, with $\wp$ = metrizable, is a generalization of A. H. Stone [13, theorem 3] as mentioned in the introduction. Also, for $\wp$ = paracompactness, theorem 3 generalizes a result of S. Hanai and A. Okuyama [2, theorem 3].A result of Y. Katuta [4, theorem 2] also follows as a corollary to Theorem 3 above.

In the end, we point out various classes of topological spaces to which our general theorems 2 and 3 are applicable. The fact that all these classes of spaces possess the property $\wp$ has been proved by different authors whose references are given below:

1. Regular	(J. Nagata [11])
2. Normal	(K. Morita [8])
3. Collectionwise normal	
4. Perfectly normal	
5. Completely-normal	
6. Metrizable	(J. Nagata [11])
7. Normal + paracompact	(K. Morita [9])
8. Regular + paracompact	(*E*. Michael [7])
9. Normal + $\mathscr{M}$-paracompact	(K. Morita [10], also see M. K. Singal and Shashi Prabha Arya [12] for a direct proof.)
10. Pointwise-paracompact	(R. E. Hodel [3])
11. Stratifiable	(J. G. Ceder [1])

Some of the known results follows from theorems 2 and 3 when applied to these classes of spaces (In particular follow theorems 5 and 6 of Y. Katuta [4]). Also these theorems give new results when applied to all the above classes of spaces.

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